"Calendar" of the *Encyclopædia Britannica*, ninth and eleventh editions. It is there shown that the year 1 of our era "had 10 for its number in the solar cycle, 2 in the lunar cycle, and 4 in the cycle of indiction"; hence "the question is therefore to find a number such, that when it is divided by the three numbers 28, 19, and 15 respectively the three quotients shall be 10, 2, and 4." Without recourse to the general discussion of Huygens it is found that the number is 4714.

Compare, F. K. Ginzel, *Handbuch der mathematischen und technischen Chronologie der Zeitrechnungsweise der Völker*, Leipsic, vol. 3, 1914, p. 182.

R. C. ARCHIBALD

SOLUTIONS.

499 (Geometry) [1916, 341; 1919, 414; 1920, 187]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the surfaces all the plane sections of which are circles.

SOLUTION BY J. L. WALSH, Harvard University.

This problem was solved by Professor Cairns [*1920*, p. 187], who interpreted the term *surface* as *algebraic surface*. Inasmuch as there exist other interesting surfaces (analytic, regular, etc.), perhaps it is worth while to give a geometric proof of the theorem:

If every plane section of a point set S is a circle, then S is a sphere.

We restrict ourselves to the real domain, and interpret our hypothesis to mean that whenever a plane actually intersects S, the intersection is a circle, which may be a null circle.

If a section of S by a sphere Σ contains three distinct points, then the section also contains the entire circle through these points. For through these three points we can pass a plane which will cut S in a circle and also cut Σ in a circle. These two circles have three points in common and hence are identical.

Let P be any point of S. Transform S into a point set S' by means of an inversion in space with P as center of inversion. The point P is transformed into P', the point at infinity; we consider as is habitual in the geometry of inversion a single point P' to lie at infinity. Every plane section of S' corresponds to a plane section or a spherical section of S. Every straight line of points belonging to S' corresponds to a circle of points belonging to S; there is no straight line all of whose points belong to S, for a straight line lies in a plane and every plane section of S is a circle.

If S' consists merely of the point P' every plane which cuts S cuts it in the null-sphere P, and the theorem is proved. If S contains another point Q besides P every plane through these points will intersect S in a circle which is not a single point, and there will be more than one of these circles. If then besides P' another point Q' belongs to S', there must be lines through Q' belonging to S', more than one of them.

Likewise, if S contains two distinct points besides P these points must lie on a circle through P, the intersection of S with the plane which contains the three points. Therefore if S' contains two distinct points besides P' it must contain the line determined by these two points.

Now we have shown that if S' contains one point Q' distinct from P' it contains more than one line through this point. It contains then the line determined by any two points of two lines through Q', and therefore all the points of at least one plane through Q'.

But a plane belonging entirely to S' corresponds in the inversion to a sphere belonging to S. Thus we have proved that if S contains more than one point it contains all the points of a sphere. S cannot contain a sphere and any point outside of the sphere, for a plane through such a point intersecting the sphere would intersect S in more than a single circle.

The method of proof used in the present note easily yields the following theorem:

If a point set consists of more than two points and is such that every spherical section which contains three points of the set also contains the circle through those three points, then the set is a circle, a plane, a sphere, or every point of space.

NOTE BY THE EDITORS.—Strictly speaking the problem as worded is more general than the theorem proved by J. L. Walsh, as it includes surfaces which are cut by a plane in more than one circle; for example, a combination of several spheres. Neither W. D. Cairns nor J. L. Walsh has solved this more general problem. Indeed there might be some further difficulty in its interpretation. For we interpret circle as including the case of a single point, and any intersection of two figures might be regarded as consisting of point circles. We might exclude such circles, but we would not want to exclude the case of a plane tangent to a sphere.

ELIJAH SWIFT gave, in effect, the same discussion as J. L. Walsh above.

2814 [1920, 134]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

The bisectors of the angles formed by the diagonals of an inscribed quadrilateral are: (1) parallel to the lines joining the midpoints of the arcs subtended by the opposite sides of the quadrilateral on its circumcircle; (2) parallel to the bisectors of the angles formed by any pair of opposite sides of the quadrilateral; (3) equally inclined to pairs of sides of the quadrilateral.

SOLUTION BY J. W. CLAWSON, Ursinus College.

Let $L_1L_2L_3L_4$ be the inscribed quadrangle. L_2L_1, L_3L_4 intersect at N'; L_1L_3, L_2L_4 at N''. D_{12}, D_{34} are the midpoints of arcs L_1L_2, L_3L_4, respectively. Let the bisector of $L_3N''L_4$ cut the circle at X; let $D_{12}D_{34}$ cut L_1L_3 at Y; let the internal and external bisectors of $L_1N'L_4$ cut L_1L_3 in W, V.

Now

$$L_3N''X = \tfrac{1}{2}L_3N''L_4 = \tfrac{1}{2}(L_3L_1L_4 + L_1L_4L_2).$$

Also

$$L_3YD_{34} = L_3D_{12}D_{34} + L_1L_3D_{12} = \tfrac{1}{2}(L_3L_1L_4 + L_1L_3L_2).$$

Again

$$L_3VN' = \frac{\pi}{2} - N'WV = \frac{\pi}{2} - \frac{1}{2}L_1N'L_3 - L_4L_3L_1 = \frac{\pi}{2} - L_4L_3L_1 - \frac{1}{2}(L_2L_1L_3 - L_4L_3L_1)$$
$$= \frac{\pi}{2} - \frac{1}{2}(L_4L_3L_1 + L_2L_1L_3).$$

But

$$L_4L_3L_1 + L_1L_3L_2 + L_2L_1L_3 + L_3L_1L_4 = \pi.$$

Hence,

$$L_3VN' = \tfrac{1}{2}(L_1L_3L_2 + L_3L_1L_4).$$

Therefore, $D_{12}D_{34}$, $N''X$, and VN' are all parallel.

Similarly, the other facts may be established.

Note. (1) was proved by Neuberg, *Mathesis*, volume 6, (1906), page 14. I believe that (2) was first published in an article of mine "The Complete Quadrilateral," *Annals of Mathematics*, volume 20 (1919), page 257. There are several other lines belonging to an inscribed quadrangle which are parallel to the bisectors of the angles between its diagonals. An account of these will be found in the paper referred to.

Also solved by F. V. MORLEY, H. L. OLSON, A. V. RICHARDSON, and ARTHUR PELLETIER.

2843 (1920, 326). Proposed by E. H. MOORE, University of Chicago.

Show that the maximum of the absolute value of $2(a + ib)(x + iy) + i(a + ib)(z + iw) + i(c + id)(x + iy)$, where $i = \sqrt{-1}$, and a, b, c, d, x, y, z, w are real numbers for which $a^2 + b^2 + c^2 + d^2 = x^2 + y^2 + z^2 + w^2 = 1$, is $1 + \sqrt{2}$. Study the locus of the point-pairs, $P = (a, b, c, d)$, $Q = (x, y, z, w)$ of the unit-sphere in real four-space for which this absolute value assumes its maximum value.